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Mathematics a Profession?

Mathematics Applied

*The Number of 12×12 Squares That Can Be Constructed by
the Method of Current Groups*

Using the Hessian to Solve a Cubic Equation

The Teacher's Department

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This Journal is dedicated to the following aims:

1. THROUGH PUBLISHED STANDARD PAPERS ON THE CULTURE ASPECTS, HUMANISM AND HISTORY OF MATHEMATICS TO DEEPEN AND TO WIDEN PUBLIC INTEREST IN ITS VALUES.
2. TO SUPPLY AN ADDITIONAL MEDIUM FOR THE PUBLICATION OF EXPOSITORY MATHEMATICAL ARTICLES.
3. TO PROMOTE MORE SCIENTIFIC METHODS OF TEACHING MATHEMATICS.
4. TO PUBLISH AND TO DISTRIBUTE TO THE GROUPS MOST INTERESTED HIGH-CLASS PAPERS OF RESEARCH QUALITY REPRESENTING ALL MATHEMATICAL FIELDS.

Mathematics a Profession ?

Followers of every calling mathematical, or largely mathematical, can never become a professional group in the sense that practitioners of the law or of medicine are a professional group until some of many ugly facts connected with our present mathematical world shall be modified or eliminated. A few of these we take courage to chronicle. The teaching of mathematics can never be professionalized,

- (1) As long as school administrations allow high school athletic coaches to teach a class in algebra or plane geometry.
- (2) As long as a mere bachelor's degree is an accepted qualification for the teaching of secondary mathematics, even if the degree represent no college mathematics whatever.
- (3) As long as there shall continue to exist a fair-sized group of college teachers of mathematics whose interest in job-holding exceeds their interest in mathematics and its teaching.
- (4) As long as a majority of those engaged in technical mathematical research remain indifferent to the steady inroads being made on the long-time tenure of mathematics in the secondary schools, though they know such inroads must ultimately cripple the very foundations of all mathematical research.
- (5) As long as vast numbers of college and university mathematicians shall continue to regard every new mathematical journal as an interloper in the field legitimate of mathematics, and as having no right to existence, instead of welcoming it as a fresh addition to the circle of journals already existing and dedicated to the glorious cause of mathematics.
- (6) As long as so many of our tribe prefer to read the history of a mathematical project that finally died and left its bones bleaching on some desert of financial sacrifice, than by sympathetic assistance to insure its immortality and enduring blessings to science.

—S. T. S.

Mathematics Applied

By E. R. SLEIGHT
Albion College

Many people consider mathematics a necessary tool, to be applied when nothing else can take its place. According to their viewpoint a student majors in this subject only if he expects to be an engineer, or if he expects to teach it. To such people mathematics is a system of hard and fast facts, without beauty or life. But when we realize the part that this subject has played in the development of our present economic, industrial and social life we must be convinced that mathematics is not dead, and that mathematicians are not relics of cloister days.

An appreciation of the place of mathematics in our every day life is gained by considering the effect of our number system. If this were suddenly to disappear, there would be utter confusion. Suppose we were compelled to use some such system as the Roman numerals. Imagine the effect upon modern business. There would be no such thing as a factory employing thousands of men. Indeed in the last analysis business is wholly dependent upon the ten number symbols. They are the foundation of computation. Ordinarily we pay little attention to them, but if we trace history of their development we find mathematicians for many centuries struggling with them in the search for a convenient notation. The symbolism of mathematics is its outstanding feature, and its real foundation stone. The struggle for proper and useful symbols began with the dawn of the subject, and is continuing even now. Perhaps nowhere do we find this better illustrated than in the calculus. The history of the development of the differential notation is as interesting as a modern detective story.

But it is not the application of such simple devices as our number system to which we desire to call attention. More and more we find business men, professional men, and men interested in such fields as economics, sociology, biology and many others turning to mathematics and mathematicians for the solutions of their difficulties. An ophthalmologist of considerable note was trying to improve his knowledge of optics. To accomplish this he found that he needed a knowledge of calculus and differential equations. It was my privilege to have this doctor in my classes three summers in succession.

Very few people know that the possibilities of wireless telegraphy had been determined by mathematicians long before Marconi announced the results of his investigation. In navigation we have one of the outstanding applications of the subject. The instruments, tables

and maps which the navigator uses are all dependent upon mathematics.

Lord Kelvin computed the strength necessary for the Atlantic cable. The engineer in charge did not follow the specifications. Result: two failures. Just recently a new epoch in sending submarine cables was begun.

By the application of certain mathematical principles it is now possible to send four times as many words in a given unit of time as has been possible in the past. Also by the application of these same principles it is now possible to transmit simultaneously more than one message over the same pair of wires.

*A huge piece of machinery was being installed in a certain factory. The man in charge of the work told the foreman of the department that best results could be obtained at so many revolutions per minute, and beyond that point there was danger to the operator. It was but a simple application of maximum and minimum theory as we find it in calculus.

In the field of engineering a well trained man is always in demand. Discoveries and inventions are no longer "cut and try" methods, but rather the application of the laws of mathematics. During the construction of the new terminal station in Cincinnati, Ohio, some difficulty arose in an attempt to bring into the station several converging lines of railroads. **One of the young engineers working on this project discovered that some of the properties of the parabola had been neglected. As soon as these mathematical facts had been applied the difficulty vanished.

It is during the past few years—since the beginning of the World War—that mathematicians have been called upon to exert themselves to the limit and to demonstrate the important position which they hold in our modern life. Certain laws of ballistics had been in use for many years, but owing to modern inventions such as long range guns, these laws were found entirely inadequate, and had to be greatly modified. Veblen of Princeton and Moulton of the University of Chicago gave their attention to the problems involved. It is interesting to note that Dr. Lapp of the University of Iowa has made use of these same laws and applied them to certain athletic activities such as throwing the javelin and punting and passing the football.

*Professor Bliss of the University of Chicago tells the story of a young salesman who came to him for help in devising a curve that would meet the demands of his chief executive. Considerable computation was involved in meeting the problem of a sale, and the chief

* Personal observation.

** One of my own students.

* Scientific Monthly, Vol. 24, page 312.

executive desired a curve that would permit the salesman to read directly from the curve the amount of the sale. Too many variables were involved in the problem, and no such curve could be devised. Professor Bliss adds that the salesman was not satisfied, and he wonders what chance a mathematician has against the demands of an executive of a large organization whose only ambition in life is a net gain for every transaction. Very often, however, it is possible for the demands to be satisfied. A graduate of one of our larger universities went into the box business with his father. He found that the salesman had considerable trouble in computing costs, and was often embarrassed in the presence of the customer. The young man appealed to the department of mathematics in the university from which he graduated, and a table was devised which reduced the time element in computing costs to a minimum. The mathematician charged \$1200 for his services, but the company was entirely satisfied as it resulted in a saving of several thousand dollars each year.

Another instance very similar to the one just cited arose in the case of the salesmen of a large woolen mill. Much time was consumed due to the fact that the unit of weight used in this mill is expressed in ounces per foot. Accordingly when the total weight in pounds and the length in yards of a blanket roll are known it is necessary to know the weight in ounces per foot. As this mill is located near Albion College, the matter was referred to the department of mathematics in that institution. A slide rule was devised which makes it possible to read the cost almost instantly.

Another problem which has been presented to the mathematician for solution is the problem of sampling. In large manufacturing concerns, turning out literally millions of duplicate pieces of one sort or another there is the question whether every article thus turned out should be inspected, or whether there is some rule of sampling which will insure against defective workmanship. A mathematician who tackles this problem is not a statistician of the sort who adds columns and takes averages, but rather this problem involves the ability of our best trained mathematicians. Charts have been formed as aids to the manufacturer, and mathematicians have been well paid since it produces a great saving to the companies.

Charts and graphs of all sorts are found in the business world. A great mail order house has a formula which connects the speed of business of the day with the amount of mail arriving in the morning.

We have noticed how great a part mathematics plays in engineering and business, and I now wish to show how this subject is ap-

plied to other fields of research and study. Physics and chemistry are fast becoming only subjects in applied mathematics. As a matter of fact some educators have even suggested that the three be combined. But we are not so familiar with the fact that the field of social science is quite dependent upon mathematical principles. In this field we find four very distinct applications—the theory of values and prices, correlation, probabilities and curve smoothing. As early as 1853 Count Curnot wrote what is now considered a classic entitled "The Mathematical Theory of Wealth." No one paid any attention to this book at the time, and only in recent years has it been appreciated. In this book the Count shows that high prices and rising prices are related to each other in exactly the same way as velocity and acceleration, and a person knowing the theory in the latter case can easily apply it to the former. Also he shows that the rate of interest ought to be higher during the period of rising prices by an amount equal to the depreciation of the dollar. Analysis shows that such has not been the case at all times, and helps explain some of our financial upheavals. *In fact Irving Fisher in an article written early in 1930 and published in the *Scientific Monthly* states that "The recent crash in the stock market is, in a large measure, the price paid for tardiness in raising the rate of interest which should have risen more than a year ago, but which was held down artificially." This statement coupled with Curnot's principle shows how essential mathematics is to economics.

Perhaps the application of mathematics to the biological sciences is not so apparent as in some other correlated fields. The usefulness of mathematics to the biologist has been recognized only in recent years. In the past biologists have taken a hostile attitude to the introduction of mathematical methods into their science. They have rather prided themselves on the fact that progress in their field was wholly observational. But such is not the case today, for they are recognizing the validity of logical deductions from given data.

Without doubt the most useful contribution made by mathematics to the biologist is the method of expressing facts in symbolical language. The progress of any science is greatly facilitated by a concise method by which it may express its results. The general idea of symbolic expression is now being adopted by biologists, and in many cases the exact mathematical symbols are used.

Then, too, chemistry and physics are being correlated with biology in such a manner as to make mathematical processes necessary. Especially is this true in biophysics and biochemistry in which subjects

* *Scientific Monthly*, Vol. 30, page 553.

the biologist is learning that his research need not be confined merely to observation, but also that cold reason may lead him to very valuable discoveries.

Mathematics is the basis of design and symmetry. The ancient Greeks and Egyptians made shapely vases, their proportions as has been recently discovered, being based on dynamic symmetry,—the more shapely ones show definite mathematical relationships between height, width, curvature and many other items. Just recently Professor Birkhoff of Harvard has developed a mathematical theory by which it is possible to test any form of art as to its accuracy and beauty. His tests are applied even to music and poetry. So we find mathematics hidden away in the most unexpected places and in most ordinary things. It is wedging itself into so many new phases of life that we feel entirely justified in teaching its principles to the students in our classes, knowing that we are preparing them to enjoy the world in which they live.

The Number of 12x12 Squares That Can Be Constructed by the Method of Current Groups

By A. L. CANDY
University of Nebraska

The purpose of this further study of Magic Squares is to get some idea of the number of approximately symmetric 12x12 squares that can be formed by the so called "Method of Current Groups," with many variations in method of selecting these "Current Groups." Starting with the 3x3 square I have first constructed six 6x6 squares. These are numbered 1, 2, 3, 4, 5, 6. In number 1, the groups are consecutive numbers (1, 2, 3, 4), (5, 6, 7, 8) etc. In number 2, the groups are not consecutive numbers, but are consecutive terms of arithmetic series whose common difference is 9. Similarly in No. 3, the common difference is 3. In Nos. 4, 5 and 6, the groups are not consecutive terms of the same series. That is, d_2 , the difference between the first and third or second and fourth terms is not equal to twice d_1 , which is the difference between the first and second, or third and fourth terms. In No. 4, $d_1=1$ and $d_2=18$. In No. 5, $d_1=1$ and $d_2=6$. In No. 6, $d_1=3$ and $d_2=18$. (See Figures in first paper).*

* See Mathematics News Letter, Vol. 8, No. 7.

Nos. 1, 2, and 3 are symmetric with respect to the center, except the end numbers in the two middle columns, and these are symmetric with respect to the horizontal axis. Nos. 4, 5 and 6 are symmetric except the end numbers in both the two middle columns and the two middle rows, and these are symmetric with respect to the horizontal axis, and the vertical axis, respectively.

From this it follows that from these 6x6 squares by the method of Consecutive Groups, using the necessary orders within the groups, we can construct 12x12 squares which shall be symmetric with respect to the center except the groups that are placed at the ends of the middle columns (in 1, 2 and 3), or the ends of the two middle rows and the two middle columns in 4, 5 and 6, and these groups will be symmetric with respect to the horizontal or vertical axis respectively.

Proceeding in this manner, I have actually written down 122 normal* squares each 12x12, of which Figure 1 is an example.

In order to construct this large number of 12x12 squares it was necessary to select the groups in 36 different ways. In only nine of these ways are the groups consecutive terms of an Arithmetic Series. The common differences (values of d) in these nine systems of groups are as follows:

$$d=1, 2, 3, 4, 6, 9, 12, 18, 36.$$

These numbers are obtained by taking the pairs of divisors of 36, the number of elements in the 6x6 square. Thus:

$$36=1 \times 36=2 \times 18=3 \times 12=4 \times 9=6 \times 6$$

The values of d_1 and d_2 in the other 27 ways of selecting the groups are given in the following table, paired according to columns.

$d_1=1$	1	1 2	1 2 3	1 3	1 2 3	4 6
$d_2=4$	6	8 8	12 12 12	18 18	24 24 24	24 24
1 2 3 6 9	1 2 3 4 6 9 12 18					
36 36 36 36 36	72 72 72 72 72 72 72 72					4

These various values of d_2 are the pairs of even divisors of 144. And the values of d_1 that can be used with any given value of d_2 are half of its even divisors.

These rules will hold for a square of any order. That is, the possible values of d are the divisors of n^2 . The possible values of

* A square in which all the groups are written in cyclic order as in Figure 1, I shall call a "Normal Square."

d^2 are all the pairs of even divisors of $4n^2$, and the possible values of d_1 are half of the even divisors of d_2 .

The groups corresponding to these values of d_1 and d_2 are easily obtained by taking corners of 2×2 squares, 3×3 squares, 4×4 squares, 7×7 squares, and corners of rectangles having these same dimensions, in the matrices—including especially the 12×12 natural square—from which the other 9 systems of groups are obtained. Only those matrices in which both dimensions are even numbers can be used for this purpose.

In all of these 122 normal squares the groups have all been written in cyclic order, with corresponding groups in each square in precisely the same order. The order used in the different columns is as follows:

**	1	2	4	3	4	3	2	1	3	4	2	1
	4	3	1	2	1	2	3	4	2	1	3	4

except that the three lower groups in each of the center columns are inverted thus

1	2	3	4
4	3	2	1

so as to be symmetric with the upper three. By this arrangement the first and second columns of groups are balanced as to rows. Also the fifth and sixth. The first and sixth columns are symmetric, as are the second and fourth. But the groups in the two middle columns are both balanced as to rows, and symmetric.

By reversing one, two, or all three of the symmetric pairs of groups in the first two and last two columns of these 6×6 squares; interchanging the two middle columns; interchanging the first two rows and also their symmetric last two rows; and also interchanging the two middle rows, 128 6×6 squares can be obtained from each of the six 6×6 squares mentioned above. In all of these changes the groups will only be reversed or inverted, and the degree of symmetry will not be affected.

By means of this transformation 128 different 12×12 normal squares can be obtained from most of the 122 squares. In a few cases, however, it is not possible to reverse the four groups—which correspond to a single group in the 6×6 square—without destroying the balance in the columns. In this way I can get 14,882 normal squares. The number at the top of each square and to the right, is the number of normal squares that can be obtained in this way.

** Figures 1, 2, 3, 4 indicate the order of the numbers in the group.

But it is not necessary to use the cyclic orders exclusively. We may use all crossed orders, or all Z orders, or various combinations of all three orders (See Figures 5, 6). Let us now consider also the number of possible combinations when all the groups in any pair of symmetric columns of a 6x6 square are all written in the same order, but not all three pairs in the same order. First, there are six possible combinations in which two such pairs have the same order; and the third pair a different order. Such a combination gives 3 permutations, for the single pair may be first and last, second and fifth, or third and fourth. Second, all three pairs of symmetric columns may be written in different orders. This combination gives 6 permutations.

4	13	104	95	36	27	126	117	128	137	46	37
13		79		33		36		8		2	
31	22	77	86	9	18	135	144	119	110	55	64
42	51	139	130	71	62	88	79	20	29	84	75
22		76		30		27		5		11	
69	60	112	121	44	53	97	106	11	2	93	102
73	82	30	21	140	131	87	78	98	107	16	7
3		9		20		23		37		25	
100	91	3	12	113	122	96	105	89	80	25	34
111	120	65	56	40	49	23	32	133	142	54	45
12		6		14		17		28		34	
138	129	38	47	67	58	14	5	124	115	63	72
43	52	143	134	39	48	92	101	24	33	85	76
26		32		10		7		27		15	
70	61	116	125	66	57	83	74	15	6	94	103
81	90	35	26	109	118	19	28	59	68	123	114
35		29		4		1		18		24	
108	99	8	17	136	127	10	1	50	41	132	141

Square No. 1

Hence, taking any given set of groups, and leaving the position of the groups unchanged, we may write the 12x12 square with the following combinations of column orders, where C stands for "Cyclic" and X stands for "Crossed":

C	C	C	C	C	C	gives	1	permutation
X	X	X	X	X	X	"	1	"
Z	Z	Z	Z	Z	Z	"	1	"
C	C	X	X	C	C	"	3	permutations
X	X	C	C	X	X	"	3	"
C	C	Z	Z	C	C	"	3	"
Z	Z	C	C	Z	Z	"	3	"
X	X	Z	Z	X	X	"	3	"
Z	Z	X	X	Z	Z	"	3	"
C	X	Z	Z	X	C	"	6	"

This gives a total of 27 squares with the same set of groups in the same positions, and the groups in any one column being in the same order, i.e. all cyclic, all crossed, etc.

Now let us consider the number of squares that can be obtained from any one of these different squares by reversing (interchanging columns), or inverting (interchanging rows) a pair, or any number of pairs of symmetric groups. All groups in cyclic order, or crossed order, are balanced as to columns. Hence, any symmetric pair of such groups, or any number of such pairs may be reversed, the square will remain magic, and symmetry will not be affected. Each pair of symmetric columns has 6 pairs of symmetric groups, each of which can be written in two ways. Hence, the 6 pairs can be written in 2^6 ways, two such pairs of columns can be written in $2^6 2^6 = 2^{12}$ ways, and three can be written 2^{18} ways.

Groups written in the Z orders are not balanced either as to columns or rows. Hence, in a column of 6 groups all in Z order, 3 must be written in one order and the other 3 must be in the reverse order. Now one in each set, or two in each set, or all three in each set, may be reversed and leave the two columns balanced. One from each set of 3 groups can be chosen in 3×3 ways, two can also be chosen in 3×3 ways, three in one way. This gives a total of 20 ways of reversing groups in a pair of symmetric columns having all groups in Z order.

Likewise, if all the groups in a row are in the same order (all cyclic, all crossed, or all Z) 3 must be inverted with respect to the other 3, in order to balance the two rows. Hence, such a row of groups may also be written in 20 different ways.

Similarly it can be shown that, if a row of groups contains a pair of groups in crossed order and balanced as to rows, or two such pairs, all other groups in the row being in either cyclic or Z order, or both, such a row may be written in 12 ways.

4	13	69	60	106	97	126	117	55	64	84	75
73		22		27		36		2		11	
31	22	42	51	79	88	135	144	46	37	93	102
77	86	139	130	36	27	53	44	128	137	11	2
79		76		33		30		8		5	
104	95	112	121	9	18	62	71	119	110	20	29
111	120	100	91	105	96	14	5	98	107	16	7
72		3		23		17		37		25	
138	129	73	82	78	87	23	32	89	80	25	34
3	12	65	56	113	122	58	67	63	72	124	115
9		6		20		74		34		28	
30	21	33	47	140	131	49	40	54	45	133	142
116	125	35	26	74	83	127	136	24	33	50	41
32		29		7		4		21		18	
143	134	8	17	101	92	118	109	15	6	59	68
43	52	108	99	1	10	57	66	94	103	123	114
26		35		7		70		75		24	
70	61	81	90	28	19	48	39	85	76	132	141

Square No. 2

Making these inversions and reversion of the groups in each of the squares given by the above combinations of orders in the three pairs of symmetric columns, we get the following results:

Column Comb.	Rows Per Inver- sions	Col. Rever- sions	Number of Squares
C C C C C C	gives $1.20^3.2^2$	$= 10^3.2^2$	$= 2,097,152,000$
X X X X X X	" $1.20^3.2^{18}$	$= 10^3.2^{21}$	$= 2,097,152,000$
Z Z Z Z Z Z	" $1.20^3.20^3$	$= 10^6.2^6$	$= 64,000,000$
C C X X C C	" $3.12^3.2^{18}$	$= 3^4.2^{24}$	$= 1,358,954,496$
X X C C X X	" $3.12^3.2^{18}$	$= 3^4.2^{24}$	$= 1,358,954,496$
C C Z Z C C	" $3.20^3.(20.2^{12})$	$= 3 \cdot 10^4.2^{16}$	$= 1,965,080,000$
Z Z C C Z Z	" $3.20^3.(20^2.2^6)$	$= 3 \cdot 10^5.2^{11}$	$= 614,400,000$
X X Z Z X X	" $3.12^3.(20.12^{12})$	$= 3^4.10 \cdot 2^{19}$	$= 424,673,280$
Z Z X X Z Z	" $3.12^3.(20^2.2^6)$	$= 3^4.10^2.2^{14}$	$= 132,710,400$
C X Z Z X C	" $6.12^3.(20.2^{12})$	$= 3^4.10 \cdot 2^{19}$	$= 424,673,280$
Total = 10,537,749,952			

Similarly, if we write all the groups in a pair of symmetric rows in the same order, but vary the order in the different pairs, we can get the same combinations of rows as the last nine above in columns. Then making the same reversions and inversions, we get from any one given square the following additional results:

Row Comb.	Rows Per Inver- sions	Col. Rever- sions	Number of Squares
C C X X C C	gives $3.20^3.2^{18}$	$= 3 \cdot 10^3.2^{21}$	$= 6,291,456,000$
X X C C X X	" $3.20^3.2^{18}$	$= 3 \cdot 10^3.2^{21}$	$= 6,291,456,000$
C C Z Z C C	" $3.20^3.2^{15}$	$= 3 \cdot 10^3.2^{18}$	$= 786,432,000$
Z Z C C Z Z	" $3.20^3.2^4$	$= 3^4.10^3.2^{12}$	$= 331,776,000$
X X Z Z X X	" $3.20^3.2^{15}$	$= 3 \cdot 10^3.2^{18}$	$= 786,432,000$
Z Z X X Z Z	" $3.20^3.2^4$	$= 3^4.10^3.2^{12}$	$= 331,776,000$
C X Z Z X C	" $6.20^3.2^{15}$	$= 3 \cdot 10^3.2^{19}$	$= 1,572,864,000$
Total = 16,392,192,000			

Taking the sum of these two totals we get

$$26,929,941,952.$$

This is the number of squares that can be obtained from a single normal 12×12 square, with a given set of groups, by using different orders for the groups in different rows and columns, and also inverting and reversing the orders in the groups, but not changing the position of a single group.

But, as has been shown above, by using different sets of groups, and placing them in different positions, we can get 14,882 different normal squares. This gives for the grand total

$$14,882 \times 26,929,941,952 = 400,771,396,129,664.$$

Allowing 9 squares to each square foot of paper, it would require about 1,590,362 square miles of paper to print all of these squares. About one-half of the area of the United States.

I wish also specially to emphasize the fact that each of the orders previously given may be rotated through an angle of ninety degrees counter clock-wise giving the following 12 additional orders in which the groups may be written:

23 32 14 41 32 23 14 41 24 42 13 31
14 41 23 32 14 41 32 23 13 31 24 42

This makes a total of 24 different ways of writing a single group.

4	31	77	104	9	36	117	144	110	137	37	64
73		79		33		36		8		2	
13	22	86	95	18	27	126	135	119	128	46	55
69	42	139	112	71	44	106	79	29	2	102	75
22		76		30		27		5		17	
60	51	130	121	62	53	97	88	20	11	93	84
73	100	3	30	113	140	105	78	107	80	34	7
3		9		20		23		37		25	
82	91	12	21	122	131	96	87	98	89	25	16
129	120	56	47	58	49	14	23	124	133	54	63
72		6		74		17		28		34	
138	111	65	38	67	40	5	32	115	142	45	72
61	52	134	125	57	48	92	83	24	15	94	85
26		32		70		7		27		15	
70	43	143	116	66	39	101	74	33	6	103	76
90	99	17	26	118	127	10	19	50	59	123	132
35		29		4		7		78		24	
81	108	8	35	109	136	1	28	41	68	114	141

Square No. 3

A few of the groups in the 6x6 squares used in forming the 12x12 squares have been written in these rotated orders, but none of the groups in the 12x12 squares have been so written, or can occur in the various transformations.

4	31	42	69	79	106	117	144	37	64	75	102
13	22	51	60	88	97	126	135	46	55	84	93
104	77	139	112	36	9	71	44	148	110	29	2
95	86	130	121	27	18	62	53	137	119	20	11
111	138	73	100	78	105	32	5	107	80	34	7
120	129	82	91	87	96	23	14	98	89	25	16
21	12	56	47	131	122	49	58	54	63	124	133
30	3	65	38	140	113	40	67	45	72	115	142
134	125	26	17	92	83	127	118	24	15	59	50
143	116	35	8	101	74	136	109	33	6	68	41
52	61	90	99	10	19	48	57	85	94	123	132
43	70	81	108	1	28	39	66	76	103	114	141

Square No. 4

These "Rotated Orders" may be utilized as follows: For the orders of the first column of the new square, top to bottom, take the orders of the first row of the given square, left to right, and rotate each order through ninety degrees counter clockwise. In like manner, get the second column from the second row, etc. In other words, rotate the entire order diagram of any given square through ninety degrees counter clock-wise, but leave the groups in precisely the same

cells of the square of order n . This process will give a new square that is balanced as to "Rows and Columns" precisely as the old square is balanced as to "Columns and Rows."

In this way we can double the number of available 6x6 squares, and hence, quadruple the possible number of 12x12 squares. (See Figures 1, 2, 3 and 4).

4	10	45	21	143	137	114	120	74	50	73	79
4		9		35		36		74		13	
34	28	15	39	119	113	138	144	56	80	55	49
16	22	33	9	132	126	101	107	85	61	86	92
10		3		30		29		19		20	
46	40	3	27	108	102	125	131	67	91	68	62
109	115	128	104	93	87	64	70	29	5	30	36
31		26		21		22		5		6	
139	133	98	122	69	63	88	94	11	35	12	6
127	121	110	134	51	57	82	76	23	47	24	18
25		32		15		76		11		12	
97	103	140	116	75	81	58	52	41	17	42	48
83	77	54	78	14	20	43	37	118	142	105	99
17		18		8		7		34		27	
53	59	84	60	38	44	19	13	136	112	123	129
96	90	65	89	2	8	31	25	106	130	117	111
24		23		2		7		28		33	
66	72	95	71	26	32	7	1	124	100	135	141

Square No. 5

The 6x6 squares used in forming these 12x12 squares are those given in figures 1, 2, 3 and 4, 5, 6 of my first paper.* In Figures 1, 2, 3, the nine groups are balanced in precisely the same way. Likewise, the groups in 4, 5, 6, are balanced in the same way, but not quite the same as in 1, 2, 3. But the groups used in 1, 2, 3, can each be balanced

* See Mathematics News Letter, Vol. 8, No. 7.

in 18 different ways thus giving 54 different squares. Likewise, the groups used in 4, 5, 6, can each be balanced in two different ways, giving 6 additional squares. We thus get a total of 60 different squares, without rotating any of the order schemes as mentioned above.

33	3	46	16	132	102	114	144	49	79	61	91
3		10		30		36		13		19	
27	9	40	22	126	108	120	138	55	73	67	85
4	34	15	45	101	131	143	113	92	62	80	50
4		9		29		35		20		14	
28	10	39	21	125	107	119	137	68	86	56	74
116	140	103	127	70	94	76	52	41	17	29	5
32		25		22		16		11		5	
110	134	97	121	64	88	82	58	47	23	36	11
133	109	122	98	87	63	57	81	24	48	12	36
31		26		21		15		12		6	
139	115	128	104	93	69	51	75	18	42	6	30
71	89	59	77	8	26	38	20	124	106	135	117
23		17		2		8		28		33	
95	65	83	53	32	2	14	44	100	130	111	141
60	78	72	90	19	37	25	7	123	105	136	118
18		24		7		1		27		34	
54	84	66	96	13	43	31	1	129	99	142	112

Square No. 6

It would be a stupendous task to substitute each of the 36 different systems of groups used in forming these 12x12 normal squares in each of these sixty 6x6 squares. I have not tried to do this. I have, however, by using only one other scheme of balancing the 6x6 squares, written down 35 additional 12x12 squares, making a total of 157 normal 12x12 squares that I have written out in full. By using these squares that I have already written and checking completely with 8 other bal-

ancing schemes, I have virtually written down 360 additional 12×12 squares. Hence, I am sure that 517 normal 12×12 squares could be written, in each of which the groups would all be the Normal Cyclic Order, as shown in Figure 1. I also estimate that the remaining 7 balancing schemes would yield about 210 more similar normal squares. Thus the grand total 768 is nearly 6 times the number of squares (122) used in the foregoing computation.

This increase in the number of normal 12×12 squares, together with the effect of rotating the order schemes (4 times) mentioned above, makes it necessary to multiply the grand total of possible 12×12 squares given above by 24, giving more than 9,600 trillions.

In all the squares so far considered, all the groups in a single column (or row) have been written in the same order. But this is not necessary. Take, for example, a square whose groups are all in cyclic order. In such a square any pair of groups in the same row, and balanced as to rows, can be changed to crossed order, provided the respective symmetric groups are also changed to crossed order. I have not tried to compute the entire number of different squares that could be obtained in this way, as it would be difficult to avoid duplications. But let us consider the square given in Figure 1. In this square one, two, three, four, or five, pairs of groups in the last two columns, and in the same row, and their symmetric groups in the first two columns, can be changed to the crossed order. Each pair of groups can be changed to crossed orders in two ways, by interchanging the first and fourth, or the second and third terms in the groups. By this process this square will give 654 different squares.

Furthermore, all the squares hitherto considered are symmetric to the same degree as the six here submitted. Let us now disregard symmetry. Then any single group that is in cyclic, or crossed order, that is not in a diagonal, or any number of such groups, can be reversed and the square will remain magic, but not symmetric. Likewise, a pair of groups in Z order in the same column, and balanced as to columns, may be reversed. Similarly, any two groups in the same line of groups, balanced as to rows, may be inverted, provided neither of the groups is in a diagonal, and the square will remain magic, but not symmetric. Thus we see that an enormous number of unsymmetric squares can also be formed by the method of Current Groups.

By rotating the "Order Diagrams" (as explained above) of these 60 normal 6×6 squares I get 120 squares. Then by interchanging the pairs of the non-symmetric numbers at the ends of the two middle columns, and at the ends of the two middle rows, I get 288 different

6x6 squares. All of these I have written down in full. Finally reversing pairs of symmetric groups that are balanced as to columns, and inverting pairs of symmetric groups that are balanced as to rows, I get as the possible number of 6x6 squares 29,568.

Using the Hessian to Solve a Cubic Equation

By JAS. A. WARD
Louisiana State University

The object of this paper is to show the relation between the roots of the cubic equation and the zeros of the Hessian of its corresponding form. This relation is used to develop a solution of the cubic in terms of the roots of the Hessian = 0.

Consider first the reduced cubic

$$(1) \quad z^3 + Pz + Q = 0, \quad P \neq 0, \quad Q \neq 0.$$

By replacing z by x/y and clearing of fractions we obtain the reduced cubic form

$$(2) \quad f = x^3 + Pxy^2 + Qy^3 = 0$$

The Hessian of this is defined to be

$$H = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix} = \begin{vmatrix} 6x & 2Py \\ 2Py & 2Px + 6Qy \end{vmatrix}$$

$$(3) \quad H = 4(3Px^2 + 9Qxy - P^2y^2)$$

Set $H = 0$, replace x/y by z , and there results what may be called the Hessian equation

$$z^2 + \frac{3Q}{P}z - \frac{P}{3} = 0$$

When compared with the general quadratic equation

$$z^2 - (r_1 + r_2)z + r_1 r_2 = 0$$

whose roots are r_1 and r_2 , we find that

$$(4) \quad P = -3r_1 r_2, \quad Q = r_1 r_2 (r_1 + r_2)$$

These values substituted in (1), (2), and (3) give

$$(5) \quad z^3 - 3r_1 r_2 z + r_1 r_2 (r_1 + r_2) = 0$$

$$(6) \quad f = x^3 - 3r_1 r_2 xy^2 + r_1 r_2 (r_1 + r_2) y^3 = 0$$

$$(7) \quad H = -36r_1 r_2 [x^2 - (r_1 + r_2)xy + r_1 r_2 y^2] \\ = -36r_1 r_2 (x - r_1 y)(x - r_2 y)$$

If $r_1 = r_2 = r$, (5) becomes

$$z^3 - 3r^2 z + 2r^3 = 0$$

It is evident that the roots of this are: $r, r, -2r$.

If $r_1 \neq r_2$, let the variable factors of (7) define new variables

$$(8) \quad \begin{cases} \mu = x - r_1 y \\ v = x - r_2 y \end{cases} \\ \therefore \begin{cases} x = \frac{r_2 \mu - r_1 v}{r_2 - r_1} \\ y = \frac{\mu - v}{r_2 - r_1} \end{cases}$$

Substitute these values in (6):

$$\frac{r_2 (r_2 - r_1)^2 \mu^3}{(r_2 - r_1)^3} - \frac{r_1 (r_2 - r_1)^2 v^3}{(r_2 - r_1)^3} = 0 \\ \therefore r_2 \mu^3 - r_1 v^3 = 0$$

$$(9) \quad \mu_1 = \sqrt[3]{(r_1/r_2)v}, \quad \mu_2 = \omega \sqrt[3]{(r_1/r_2)v}, \quad \mu_3 = \omega^2 \sqrt[3]{(r_1/r_2)v}, \text{ where } \sqrt[3]{(r_1/r_2)}$$

is the cube root with the smallest positive amplitude and ω is a primitive cube root of unity. Substitute (8) in (9) and replace x/y by z :

$$z_1 = -\sqrt[3]{r_1^2 r_2} - \sqrt[3]{r_1 r_2^2}$$

$$(10) \quad \begin{aligned} z_2 &= -\omega \sqrt[3]{r_1^2 r_2} - \omega^2 \sqrt[3]{r_1 r_2^2} \\ z_3 &= \omega^2 \sqrt[3]{r_1^2 r_2} - \omega \sqrt[3]{r_1 r_2^2} \end{aligned}$$

where the roots are so chosen that the product of the terms in the right member of each of the above is $r_1 r_2$. Thus we have the solution of the reduced cubic in terms of the roots of its Hessian equation.

By adding d to each of the roots in (10) we get

$$(11) \quad \begin{aligned} z_1 &= -\sqrt[3]{r_1^2 r_2} - \sqrt[3]{r_1 r_2^2} + d \\ z_2 &= -\omega \sqrt[3]{r_1^2 r_2} - \omega^2 \sqrt[3]{r_1 r_2^2} + d \\ z_3 &= -\omega^2 \sqrt[3]{r_1^2 r_2} - \omega \sqrt[3]{r_1 r_2^2} + d \end{aligned}$$

The equation with these roots is

$$(12) \quad z^3 - 3dz^2 + 3(d^2 - r_1 r_2)z + (r_1^2 r_2 + r_1 r_2^2 + 3r_1 r_2 d - d^3) = 0$$

which is the same as the general cubic

$$(13) \quad z^3 + Bz^2 + Cz + D = 0$$

in which

$$(14) \quad B = -3d, \text{ etc.}$$

Proceeding with (12) as with (1) we find that the Hessian equation is

$$z^2 - (r_1 + r_2 + 2d)z + (r_1 r_2 + r_1 d + r_2 d + d^2) = 0$$

whose roots are

$$(15) \quad s_1 = r_1 + d, s_2 = r_2 + d; \text{ or } s_1 = r_1 - B/3, s_2 = r_2 - B/3$$

Adding d to the roots of the cubic adds d to the roots of its Hessian equation also.

If $r_1 = r_2 = r$, $s_1 = s_2 = s = r + d$, and (12) becomes

$$z^3 - 3dz^2 + 3(d^2 - r^2)z + (2r^3 + 3r^2 d - d^3) = 0$$

The roots of this are easily verified to be

$$(16) \quad r + d, r + d, -2r + d; \text{ or } s, s, -2s + B$$

In (1) if $P = 0$, we get $z^3 + Q = 0$

By adding d to each of its roots we obtain

$$(17) \quad z^3 - 3dz^2 + 3d^2z + Q - d^3 = 0$$

By the same method as before we find that its Hessian equation is

$$Q(z-d)=0$$

If $Q \neq 0$, this is linear and the roots of (17) are

$$(18) \quad \sqrt[3]{-Q+d}, \omega \sqrt[3]{-Q+d}, \omega^2 \sqrt[3]{-Q+d}$$

If $Q=0$, the Hessian vanishes and the roots of (17) are all equal and are

$$(19) \quad z=d, \text{ or } -B/3$$

Summarizing, we find that we can write down the roots of the general cubic if we first find the roots of its Hessian equation, a quadratic (at most) that can always be solved. We proceed as follows: Given the cubic

$$z^3+Bz^2+Cz+D=0,$$

find its Hessian.

Case 1. If $H=0$, the roots of the cubic are all equal:

$$z = -B/3$$

Case 2. If H is linear, the roots of the cubic are

$$\sqrt[3]{-D+(B/3)^3-B/3}, \omega \sqrt[3]{-D+(B/3)^3-B/3}, \omega^2 \sqrt[3]{-D+(B/3)^3-B/3}$$

Case 3. If H is quadratic:

A. If the roots of $H=0$ are equal: $S_1=S_2 \equiv s$,

then the roots of the cubic are

$$s, s, -2s+B.$$

B. If the roots of $H=0$ are unequal, the roots of the cubic are

$$z_1 = -\sqrt[3]{r_1^2 r_2} - \sqrt[3]{r_1 r_2^2} - B/3$$

$$z_2 = -\omega \sqrt[3]{r_1^2 r_2} - \omega^2 \sqrt[3]{r_1 r_2^2} - B/3$$

$$z_3 = -\omega^2 \sqrt[3]{r_1^2 r_2} - \omega \sqrt[3]{r_1 r_2^2} - B/3$$

Thus we have a solution of the cubic that is very different from Cardan's and also easier. In numerical cases it is a real time saver, especially because the general equation does not have to be reduced first and then the special cases are found in the very beginning.

It is well to note that the irreducible case is included in Case 3.B., for we cannot algebraically extract the cube root of a complex number.

We now give some numerical illustrations of this method:

$$\begin{aligned}
 (1) \quad & z^3 + 6z^2 + 12z + 8 = 0 \\
 & f = x^3 + 6x^2y + 12xy^2 + 8y^3 = 0 \\
 & H = 72 \begin{vmatrix} x+2y & 2x+4y \\ x+2y & 2x+4y \end{vmatrix} = 0
 \end{aligned}$$

\therefore the roots of the cubic are $-6/3 = -2$ by Case 1.

$$\begin{aligned}
 (2) \quad & z^3 + 6z^2 + 12z + 35 = 0 \\
 & f = x^3 + 6x^2y + 12xy^2 + 35y^3 = 0 \\
 & H = 972(x+2y) \\
 & z+2=0, \quad z = -2
 \end{aligned}$$

By case 2, the roots are:

$$\begin{aligned}
 z_1 &= \sqrt[3]{-35 + (6/3)^3} - 6/3 = -3 - 2 = -5 \\
 z_2 &= -3\omega - 2 \\
 z_3 &= -3\omega^2 - 2
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & z^3 - 6z^2 + 15z + 100 = 0 \\
 & f = x^3 - 6x^2y + 15xy^2 + 100y^3 = 0 \\
 & H = -324(x^2 - 10xy + 25y^2) \\
 & z^2 - 10z + 25 = 0 \\
 & z = 5 = s
 \end{aligned}$$

By case 3.A. the roots of the cubic are

$$5, 5, \text{ and } -4$$

$$\begin{aligned}
 (4) \quad & z^3 + 6z^2 + 3z + 18 = 0 \\
 & f = x^3 + 6x^2y + 3xy^2 + 18y^3 = 0 \\
 & H = 36(-3x^2 + 16xy + 35y^2) \\
 & 3z^2 - 16z - 35 = 0 \\
 & s_1 = 7, \quad s_2 = -5/3
 \end{aligned}$$

$$r_1 = 7 + 2 = 9; \quad r_2 = -5/3 + 2 = 1/3 \text{ by (15)}$$

Hence the roots are, by Case 2.B.:

$$z_1 = -\sqrt[3]{81/3} - \sqrt[3]{9/9} - 2 = -3 - 1 - 2 = -6$$

$$z_2 = -3\omega - \omega^2 - 2 = \sqrt[3]{-3}$$

$$z_3 = -3\omega^2 - \omega - 2 = -\sqrt[3]{-3}.$$

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Purchasers of back volumes should apply to G. Bell & Sons, Ltd., Portugal St., W. C. 2., London, England. —L. J. ADAMS.



The Teacher's Department

Edited by

JOSEPH SEIDLIN AND W. PAUL WEBBER



IN REVIEW

It may be fitting that this, the final issue of Volume 9 of the NATIONAL MATHEMATICS MAGAZINE, carry in its Teachers Department some comment by Professor Webber and myself on the first year of this department.

The criticisms that have reached me with reference to the magazine in general and the Teachers Department in particular, have often been severely adverse to many procedures, opinions, and content, but always commendatory of the idea of a Teachers Department. Both for the Teachers Department and for the cause of teaching mathematics, I regard this expression of genuine approval a very encouraging sign, indeed.

I do not feel the same degree of optimism with reference to contributions. It has been, and still is, a difficult matter to get expressions of opinions from many of our colleagues, who both have opinions and know how to express them. Aside from busyness, genuine preoccupation with their own curricula and extra-curricula activities, (and a trace of laziness common to the craft), there seems to be a sense of timidity.

Men, and women, generally recognized as fine teachers, who express themselves often and freely not only in matters intimately related to their field of research but even on the relative merits of diverse systems in contract bridge, feel entirely too restrained to share with their colleagues the fruit of their teaching ability and experiences.

Much of the little that has been written on the teaching of college mathematics is in the nature of a lament on the present state of affairs. Too true, some of the observed teaching and many of the observed and known results of such teaching are deplorably poor. But we have many good, some excellent, and a few brilliant teachers. We have teachers who inspire the gifted students and teachers who encourage the mediocre; teachers who attract and develop the "torch bearers" and teachers who increase the number and zest of the "true followers."

It is hoped that the pages of the Teachers Department may help spread the influence of these excellent and gifted teachers. It is not necessary that these people write dissertations or make retiring presi-

dential or vice-presidential addresses. Occasionally, a pointed paragraph may prove to be of inestimable value in the cause of better teaching. Recently I received a gem from Professor H. W. Tyler, whom the teaching profession lost and the American Association of University Professors "gained," when he retired as professor and head of the department of mathematics at the Massachusetts Institute of Technology some years ago. I quote from Professor Tyler's letter to me:

"One of the major difficulties of students in mathematics is to acquire a measure of unified perspective of the various subjects. They learn the lessons from day to day, but often fail to understand 'what it is all about.' The text books do not and, I think, cannot meet this need so adequately as a competent teacher. This is, in my judgment, the main reason why the teacher should utilize a fraction of his time, preferably at the beginning of the exercise, in giving a sort of bird's-eye view, not at every exercise, but at the beginning of the course and whenever a new division of the subject is entered. It seems to me particularly worth while to do the same sort of thing, but from a different angle, in a few review exercises at the end of a course, not as a matter of preparation for an examination, but as a way of integrating the student's understanding of the subject. If this seems sinful pedagogy to any of our straight-laced friends, I should be inclined to say, 'What of it?'. Sometimes there is need to explain the advance lesson per se. Such explanation should be carefully restricted to guidance actually needed, against waste of the student's time. It should never make it easy or possible for even the bright students to escape making their own preparation."

I do not mean to imply that every good teacher need agree wholly with Dr. Tyler, but isn't it obviously desirable that every teacher of elementary college mathematics know of the way in which an excellent teacher of much and varied experience handles the vital problem of the "advance lesson?"

It is not a question of authority or reputation or a name. It is sharing one's experiences with the profession; it is improving the teaching of mathematics; it may be the beginning of a science and a philosophy of the teaching of elementary college mathematics.

Below is an announcement of the forthcoming courses, in a sense the first of their kind, on the teaching of college mathematics. Our praise goes to Teachers College, Columbia University; our best wishes to Professor Hedrick; and our exhortations to attend these courses to all live and progressive young and old teachers of college mathematics.

—JOSEPH SEIDLIN.

A TEACHER'S DEPARTMENT IN A MATHEMATICAL JOURNAL

In general, high school work has come to be concerned with innocuous specific mental reactions to such a degree, that the name of a course in mathematics carries little assurance that the work done is actually mathematical in spirit. In general, colleges must accept what the high schools produce. The public will demand this in the long run. The lack of serious purpose and of the probability of obtaining any considerable set of distinctly useful mathematical habits from such high school instruction is not hard to realize when we consider the prevailing type of questions insisted upon in many quarters. We annex a sample of such tests, the making of which has become such an art as to be almost regarded as a separate profession. These are patterned after those of a long list that appeared in a teacher's journal.

1. If two triangles have their respective angles equal, what can be proved about their sides?
2. In what kind of triangles is the square on one side equal to the sum of the squares on the other two sides?
3. The opposite sides of a parallelogram are parallel. Yes, No, Neither. Check the correct answer.
4. The opposite sides of a parallelogram are perpendicular. Yes, No, Neither. Check the correct answer.
5. A quadrilateral is a plane figure having () sides. Complete the statement by filling in one or more words, etc.

In a list of probably two hundred questions there does not appear an actual mathematical problem or theorem. The test is to be answered from mere memory without the exercise of consecutive thinking processes. There is required no span of attention beyond that of a mere child. Such quizzes are said to be objective, and this word has become almost a fetish in many places. College professors are sometimes called upon to conduct quizzes in high school contests. They are usually advised that the test should be as objective as possible. Is it difficult to see why there is a jolt for the pupils when they enter college or undertake any work requiring serious knowledge? Education, as is the case in other human affairs, has been veered too far from the main track and as a consequence has lost much that is vital. It is the business of those concerned to try to bring about a better state of affairs.

The year just closing has been one of concern for the editors of the Teacher's Department of this magazine. We have taken our task seriously. There have been encouraging responses and comments. There

have been comments not so complimentary. Probably this is to be expected. We are still of the opinion that there is much to be done in this field. The difficulty is to obtain such material as is needed and when needed. We presume that every good cause must have the same experience. We wish to thank those who have contributed and we desire them to continue to contribute. We desire other contributions in order to get as complete a cross section of teaching problems and methods as possible for our readers.

For some years teachers of mathematics in colleges have been expected to take less and less for granted on the part of our pupils in actual substantial preparation. This must require greater attention to teaching methods and to course content in order to reach a minimum desirable status. Let us have a whole hearted cooperation and a progressive attitude in our work so that we may perform some of the needed service in education with regard to mathematics. Our problem may go further than just teaching pupils. There is need for the cultivation of better understandings and greater frankness without offensiveness in order that mathematics may serve in its rightful place in our educational system. The remedy for our present condition is not in elimination, or further sterilization of mathematics courses, but in more cooperation among ourselves and with school administrations in general. —W. PAUL WEBBER.

Dr. E. R. Hedrick will give two courses at Columbia University this summer. One will be on professionalized subject matter in algebra and geometry, in which he will treat those topics in elementary algebra and geometry which offer peculiar difficulties to teachers and on which many teachers desire to be further informed, such as the foundations of mathematics, the number system of algebra, function concepts, and the like. A knowledge of the calculus is presupposed. He will give a second course on teaching mathematics in junior colleges and in lower divisions of colleges and universities. Here an attempt will be made to study the pedagogical questions that arise in instruction in college algebra, trigonometry, analytic geometry and the calculus. Various ways of arranging subject matter will be discussed, together with some study of the relationship of mathematics to the other fields of knowledge such as physics, engineering, and the like.



Notes and News

Edited by
I. MAIZLISH



In association with the annual meeting of the Louisiana Academy of Sciences the Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics and the Louisiana-Mississippi Section of the Mathematical Association of America held their annual joint meeting at Pineville, Louisiana on March 29th and 30th. Professor H. J. Ettlinger of the University of Texas delivered the main address of the three joint conventions, his topic being "Mathematics as an Experimental Science."

At the Friday afternoon meeting of the Council Branch the following papers were read:

"Husbandry in Mathematics"—J. T. Harwell, Byrd H. S., Shreveport, La.

"A Simple Construction for the Approximate Trisection of an Acute Angle"—J. R. Hitt, Misissippi College.

"The Hessian Solution of the Cubic Equation"—Jas. A. Ward, L. S. U.

"The Correlation of Public School Mathematics"—V. K. Fletcher, Selma (La.) H. S.

Report of the Committee on "The High School Graduate and the College Teacher of Mathematics"—P. K. Smith, L. P. I.

The Section program carried out on Saturday morning included the following:

"Groups of Space Transformation Resulting from Inversions in Spheres"—May Hickey, Delta State Teacher's College.

"The Geometry of the Complex Triangle"—B. E. Mitchell, Mill-saps College.

"Some problems solved by Heaviside's Directional Calculus"—J. F. Thomson, Tulane University.

"A Construction for the Tangents at the Nodes of the Rational Plane Quintic"—Elsie J. McFarland, Jones Co. (Miss.) Junior College.

"A Foundation for Riemannian Geometry"—H. L. Smith, L. S. U.

"Non-unique Solutions of First Order Ordinary Differential Equations"—H. J. Ettlinger, University of Texas.

The business meeting of the Louisiana-Mississippi section of M. A. of A. elected as official for next year

H. L. Smith, L. S. U., chairman.

Dorothy McCoy, Belhaven College, Secretary-treasurer.

V. B. Temple, Louisiana College, Vice chairman for Louisiana.

D. S. Dearman, State Teachers College, Hattiesburg, Miss. Vice-chairman for Mississippi.

Hattiesburg, Mississippi will be the next meeting place.

FOREIGN MATHEMATICS JOURNALS

1. Norsk Matematisk Tidsskrift Norway
2. Gazeta Matematica Bucharest
3. The Journal of the Indian Mathematical Society Madras
4. Jahrbuch Über Die Fortschritte Der Mathematik Berlin
5. Bulletin de la Societe Mathematique de Grece Athens
6. L'Enseignement Mathematique Paris
7. Matematica Elemental Madrid
8. Revista Matematica Hispano-Americana Madrid
9. Casopis Matematiky a Fysiky Prague
10. The Tokio Mathematical Journal Sendai, Japan
11. Tableau des Sciences physiques et mathematiques en
Belgique Belgique
12. Journal fur die reine und angewandte Mathematik Berlin

L. J. ADAMS.

ARISTOTLE ON THE CONCEPT OF CONTINUITY

The man who systematized deductive logic must be admitted to have performed a great service to geometry. But Aristotle's benefits are not confined to this. He is the author or the improver of many of the most difficult geometrical definitions. One of these may be here given. He defined continuity as follows: "A thing is continuous when of any two successive parts the limits, at which they touch, are one and the same, and are, as the word implies, held together." Hence, he said in answer to Zeno, motion is not, like counting, a discrete operation, a series of jerks: the moved thing does not stop at the stages which the calculator chooses to make.

From "A Short History of Greek Mathematics," by JAMES GOW.



Problem Department

Edited by
T. A. BICKERSTAFF



This department aims to provide problems of varying degrees of difficulty which will interest anyone who is engaged in the study of mathematics.

All readers, whether subscribers or not, are invited to propose problems and to solve problems here proposed.

Problems and solutions will be credited to their authors.

While it is our aim to publish problems of most interest to the readers, it is believed that regular text-book problems are, as a rule, less interesting than others. Therefore, other problems will be given preference when the space for problems is limited.

Send all communications about problems to T. A. Bickerstaff, University, Mississippi.

LATE SOLUTIONS

No. 79. By A. C. Briggs, Dewey C. Duncan, and J. K. R. Stauffer.

No. 78. By A. C. Briggs, E. N. Yeager, D. C. Duncan, J. R. K. Stauffer, J. W. Foust.

No. 75. Proposed by Alexander Y. Boldyreff, University of Arizona.

Show that the continued radical

$$(1) \quad \sqrt[r]{PQ + (P^{r-1} - Q)\sqrt[r]{PQ + (P^{r-1} - Q)\sqrt[r]{PQ + \dots \text{to } \infty}}}$$

is equal to P , where P, Q are positive and r is a positive integer > 1 .

Solution by Theodore Bennett, University of Wisconsin.

To study the expression (1) we set $Q = uP^{r-1}$, so that the expression becomes

$$(2) \quad P \sqrt[r]{u + (1-u)\sqrt[r]{u + (1-u)\sqrt[r]{u + \dots \text{to } \infty}}}$$

Before finding the "value" of this expression by formal operations we must show that such a value exists. Let us set

$$(3) \quad f(u, n) = \sqrt[r]{u + (1-u)\sqrt[r]{u + \dots \text{to } n}}$$

where we include only the first n radicals; i. e. $Pf(u, n)$ is the expression (2) with the $(n+1)$ -st radical replaced by zero. The "value" of (2) is then *defined* to be the limit (if it exists) of $Pf(u, n)$ as $n \rightarrow \infty$.

We consider only the special case in which $0 < u < 1$. From the definition of $f(u, n)$ we have

$$(4) \quad f(u, n+1) = \sqrt[r]{u + (1-u)f(u, n)};$$

therefore

$$(5) \quad f(u, n+1) - f(u, n) = (1-u)[f(u, n) - f(u, n-1)].$$

Since $1-u > 0$ it follows from (5) that $f(u, n+1) > f(u, n)$ provided $f(u, n) > f(u, n-1)$; but it is easily seen that $f(u, 2) > f(u, 1)$, so that by induction we have $f(u, n+1) > f(u, n)$ for every n .

If in (4) we set $1-u = \epsilon$, where $\epsilon > 0$, we have

$$f(u, n+1) = \sqrt[r]{1-\epsilon + \epsilon f(u, n)} = \sqrt[r]{1-\epsilon(1-f(u, n))}.$$

This shows that $0 < f(u, n+1) < 1$ provided this is true of $f(u, n)$; but this is evidently true of $f(u, 1)$, and hence by induction it is true for every n .

Therefore the sequence $f(u, n)$ for $n = 1, 2, 3, \dots$ is monotonic increasing and bounded, whence $f(u, n)$ approaches a limit as $n \rightarrow \infty$. If this limit be x , (4) gives

$$x = \sqrt[r]{u + (1-u)x},$$

whence x is the real root of

$$(x-1)(x^{r-1} + x^{r-2} + \dots + x + u) = 0,$$

which gives $x = 1$. Hence (2) has the value P .

If r is even and $u > 1$ some of the terms of the sequence $f(u, n)$ may be imaginary, but not necessarily so; even in this case it is possible for the limit to exist and be real. If r is odd with $u > 1$ the terms are real, but do not form a monotonic sequence. I suspect that the sequences formed by the even terms and the odd terms are each monotonic. Probably a modification of the method used above could be used in all cases in which the terms are real.

No. 77. Proposed by Richard A. Miller, University of Mississippi.

A car is traveling at the rate of 60 miles per hour along a level road when a "lug" is thrown from the rear wheel after making one-sixth revolution from the lowest point. If the diameter of the wheel

is 30 inches and the lug is 6 inches from the outer circumference, determine the motion, the range, and the maximum height of the lug disregarding air resistance. Investigate the case where the road is inclined at an angle θ from the horizontal.

Solution by James A. Ward, L. S. U.

The parametric equations of the lug's path are:

$$(1) \quad y = a_0 + V_0 t \sin \alpha - gt^2/2$$

$$(2) \quad x = (V_c + V_0 \cos \alpha)t$$

where a_0 is the initial height of the lug, V_0 is the initial velocity of the lug due to the rotation of the wheel, α is the angle of elevation of the velocity V_0 , g is the acceleration of gravity, V_c is the velocity of the car.

When y is a maximum, $y' = 0$

$$y' = V_0 \sin \alpha - gt = 0$$

$$t_m = V_0 \sin \alpha / g$$

$$(3) \quad y_m = a_0 + (V_0 \sin \alpha)^2 / 2g$$

$$V_c = 60 \text{ miles per hour} = 88 \text{ feet per second.}$$

The angular velocity of the wheel is

$$\omega = (88/\pi d)2\pi = 70.4 \text{ radians per second.}$$

$V_0 = r\omega$, where r is the distance from the center of the wheel to the lug.

$$V_0 = (70.4)3/4 = 52.8 \text{ feet per second.}$$

The lug is thrown from the rear wheel after making one sixth revolution from the lowest point. Hence the wheel has revolved through 60° and the tangent to the wheel at that point makes an angle of 120° with the direction of motion of the car. Hence $\alpha = 120^\circ$.

The lug is 6 inches from the ground at the lowest point. When the wheel revolves through 60° , the lug rises $9 \sin (90 - 60)^\circ = 4\frac{1}{2}$ inches, which makes $a_0 = 10\frac{1}{2}$ inches or $\frac{7}{8}$ foot.

Take $g = 32.2$.

With these values for the constants, it is found from (3) that the maximum height of the lug is

$$(4) \quad y_m = \frac{7}{8} + (52.8 \sin 120)^2 / 2(32.2) = 33.34 \text{ feet}$$

The lug will strike the ground when $y=0$ in equation (1).

$$.875 + 52.8 t \sin \alpha - 16.1 t^2 = 0$$

$$t = 2.85 \text{ seconds}$$

To find the range substitute this value of t in equation (2).

$$x = 175.56 \text{ feet, the range of the lug.}$$

If the car is on a road inclined at an angle θ , take θ positive if the car is going uphill. The parametric equations of the lug now become:

$$(1') \quad y = a_0 \sec \theta + (V_c \sin \theta + V_0 \sin \varphi)t - gt^2/2$$

$$(2') \quad x = (V_c \cos \theta + V_0 \cos \varphi)t$$

where $\varphi = \theta + \alpha$. These equations become the same as (1) and (2) if $\theta = 0$.

y is a maximum when $y' = 0$

$$y' = V_c \sin \theta + V_0 \sin \varphi - gt = 0$$

$$t_m = (V_c \sin \theta + V_0 \sin \varphi)/g$$

$$(3') \quad y_m = a_0 \sec \theta + (V_c \sin \theta + V_0 \sin \varphi)^2/2g$$

The parametric equations of the road are:

$$(4') \quad y = u \sin \theta$$

$$(5') \quad x = u \cos \theta$$

When the lug strikes the ground, the values of x given in equations (5') and (2') are equal, and the values of y given in (4') and (1') are also equal.

$$x = u \cos \theta = (V_c \cos \theta + V_0 \cos \varphi)t$$

$$u = (V_c \cos \theta + V_0 \cos \varphi)t/\cos \theta$$

$y = (V_c \sin \theta + V_0 \cos \varphi \tan \theta)t = a_0 \sec \theta + (V_c \sin \theta + V_0 \sin \varphi)t - gt^2/2$
from which it is seen that

$$(6') \quad gt^2/2 - V_0 t \sin \alpha \sec \theta - a_0 \sec \theta = 0$$

The value of t found here is the total time the lug was in the air. Substitute this in (2') to obtain the horizontal distance the lug travels,

and substitute this value in (5') to obtain the distance u , which is the distance along the road.

Also solved by A. C. Briggs, and the proposer.

No. 78. A solution of No. 78 by Theodore Bennett, University of Wisconsin.

The essential part of the problem is to evaluate the expression

$$(1) \quad \sqrt{u + \sqrt{u + \sqrt{u + \sqrt{u + \dots}}}} \text{ to } \infty,$$

where u is a positive real number. Let us set

$$(2) \quad f(u, n) = \sqrt{u + \sqrt{u + \sqrt{u + \dots}}}$$

in which only the first n radicals are present; i. e. $f(u, n)$ is the expression (1) with the $(n+1)$ -st radical replaced by zero. Then we have

$$(3) \quad f(u, n+1) = \sqrt{u + f(u, n)}.$$

Since

$$(4) \quad f^2(u, n+1) - f^2(u, n) = f(u, n) - f(u, n-1)$$

we see that $f(u, n+1) > f(u, n)$ provided $f(u, n) > f(u, n-1)$; but obviously $f(u, 2) > f(u, 1)$, so that by induction $f(u, n+1) > f(u, n)$ for every n .

We now show that $f(u, n)$ is less than $1+u$. Equation (3) gives

$$(5) \quad (1+u)^2 - f^2(u, n+1) = (1+u)^2 - u - f(u, n) \\ = u^2 + (1+u - f(u, n)).$$

This shows that $1+u - f(u, n+1) > 0$ provided this is true of $f(u, n)$; but it is evidently true of $f(u, 1)$, so that by induction it is true for every n . Hence the sequence $f(u, n)$ for $n=1, 2, 3, \dots$ is monotonic increasing and bounded, so that $f(u, n)$ approaches a limit as $n \rightarrow \infty$.

If this limit be x , (3) gives

$$(6) \quad x = \sqrt{u + x},$$

whence

$$(7) \quad x = \frac{1}{2} + \sqrt{u + \frac{1}{4}}$$

is the value of (1).

Nos. 75 and 78. Attention is called to unwarranted assumptions made in the solutions of these problems in the April number of this magazine, by H. L. Smith, L. S. U.; Theodore Bennett, University of Wisconsin, and Aaron Herschfeld, Brooklyn, N. Y.

The latter two offer some suggestions and remarks which are given place here.

Remarks concerning 75 by Aaron Herschfeld:

In order to prove convergent the continued radicals involved we write

$$\begin{aligned} & \sqrt[m]{PQ + (P^{m-1} - Q)\sqrt[m]{PQ + (P^{m-1} - Q)\sqrt[m]{\dots}}} \\ & = \sqrt[m]{A + B\sqrt[m]{A + B\sqrt[m]{A + \dots}}} \end{aligned}$$

Where

$$A = PQ \text{ and } B = P^{m-1} - Q$$

Now,

$$\begin{aligned} & \sqrt[m]{A + B\sqrt[m]{A + B\sqrt[m]{A + B\sqrt[m]{A + \dots}}}} \\ & = B^{1/m-1} \sqrt[m]{A/B^{m-1} + \sqrt[m]{A/B^{m-1} + \sqrt[m]{\dots}}} \end{aligned}$$

It can be proved that $\sqrt[m]{x + \sqrt[m]{x + \sqrt[m]{x + \dots}}}$ Converges if $m > 1$ and $x > 0$.

(See Amer. Math. Monthly, Jan., 1917, Problem 460.)

Hence our original radical is convergent if $A/B^{m-1} > 0$ or if $PQ > 0$, $P^{m-1} > Q$

Concerning No. 78: Here the question of the convergence of

$$\sqrt{n + \sqrt{n + \sqrt{n + \dots}}}$$

is involved and is therefore already disposed of.

No. 80. Proposed by Donat Kazarinoff, University of Michigan.

Consider the sections of a paraboloid of revolution by planes through its focus. Show that the focus of the surface is a focus of every section and the corresponding directrices lie in a plane.

Solved by J. K. R. Stauffer, Laurel, Del.

Let the paraboloid of revolution with its focus at the origin be represented by

$$x^2 + y^2 = p(2z + p).$$

I

The planes passing through the focus may be represented by

$$\lambda_3 x + \mu_3 y + \nu_3 z = 0, \quad (\lambda_3^2 + \mu_3^2 + \nu_3^2 = 1) \quad \text{II}$$

In order to rotate the axes so that this plane is one of the coordinate planes, put

$$\begin{aligned} x &= \lambda_1 x' + \lambda_2 y' + \lambda_3 z' \\ y &= \mu_1 x' + \mu_2 y' + \mu_3 z' \quad \lambda_i \lambda_j + \mu_i \mu_j + \nu_i \nu_j = \begin{matrix} 0 & i \neq j \\ 1 & i = j \end{matrix} \\ z &= \nu_1 x' + \nu_2 y' + \nu_3 z' \end{aligned} \quad \text{III}$$

Equations I and II become

$$(1 - \nu_1^2)x'^2 + (1 - \nu_2^2)y'^2 + (1 - \nu_3^2)z'^2 - \text{IV}$$

$$2\nu_1\nu_2x'y' - 2\nu_1\nu_3x'z' - 2\nu_2\nu_3y'z' = p(2\nu_1x' + 2\nu_2y' + 2\nu_3z' + p)$$

and $z' = 0$, respectively. The section of the paraboloid of revolution formed by the plane through its focus is found by putting $z' = 0$ in IV. The equation of the conic in the plane $z' = 0$ is

$$(1 - \nu_1^2)x'^2 + (1 - \nu_2^2)y'^2 - 2\nu_1\nu_2x'y' - 2p(\nu_1x' + \nu_2y') = p^2. \quad \text{V}$$

In the plane $z' = 0$, rotate the axes by means of the transformation

$$x' = (\nu_1^2 + \nu_2^2)^{1/2}(\nu_1x'' - \nu_2y'') \quad y' = (\nu_1^2 + \nu_2^2)^{1/2}(\nu_1x'' + \nu_2y'') \quad \text{VI}$$

The equation of the conic then becomes

$$\nu_3^2x''^2 + y''^2 - 2p(1 - \nu_3^2)^{1/2}x'' = p^2 \quad \text{VII}$$

The equation VII may be put in the form

$$\begin{aligned} b^2(x - c)^2 + a^2y^2 &= a^2b^2 \\ a &= p/\nu_3^2, \quad b = p/\nu_3, \quad c = p(1 - \nu_3^2)^{1/2}/\nu_3^2 \end{aligned}$$

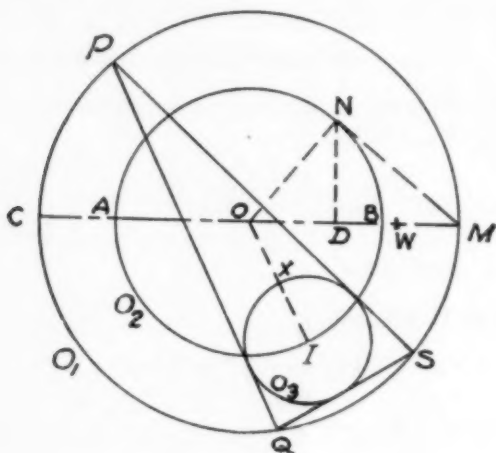
It is seen that the conic for every plane given by I is, in general, ($\nu_3 \neq 0, 1$) an ellipse whose focus is at the origin. Since the origin remains fixed under the two rotations one of the foci of the ellipse coincides with the focus of the paraboloid of revolution.

The directrix of the ellipse in the $z' = 0$ plane corresponding to the considered focus is given by

$$x'' = c - a^2/c \text{ or } x'' = -p/(1 - \nu_3^2)^{1/2}$$

Using the inverses of III and VI, it is found that the locus of the directrices is the plane $z = -p$.

Also solved by A. C. Briggs.



No. 84. Proposed by Walter B. Clarke, San Jose, Calif.

Given two concentric circles O_1 and O_2 , the outer one being O_1 . Required to find circle O_3 with center on O_2 and such that if from any point P on O_1 tangents are drawn to O_3 and cutting O_1 at Q and S , the line QS will also be tangent to O_3 .

Solution by R. A. Miller, University of Mississippi.

Analysis: The required circle O_3 (center I) will be incircle to PQS , of which O_1 is circumcircle. One locus of I is O_2 . The distance between the circumcenter and incenter, OI , is known—the radius of O_2 . Let $OM = R$, $IX = r$, $OI = d = ON$, $O =$ center of O_1, O_2 . Hence it is required to find inradius, r , from the Euler Theorem:

$$OI^2 = d^2 = R(R - 2r).$$

$$r = \frac{R^2 - d^2}{2R} = \frac{(R - d)(R + d)}{2R} = \frac{(OM - OB)(OM + OB)}{2 OM} = \frac{MB \cdot MA}{2 OM}$$

$$MB \cdot MA = MN^2 \quad (\text{Square of tangent equal product segments of secant})$$

$$MN^2 = OM^2 - ON^2 \quad ON^2 = ND^2 + OD^2. \quad (\text{Rt. triangle relations.})$$

$$ND^2 = OD \cdot MD \quad (\text{Square of alt. to hypotenuse = product of segments of hyp})$$

$$\begin{aligned}
 \text{Hence } r &= \frac{MN^2}{2 OM} = \frac{OM^2 - ON^2}{2 OM} = \frac{OM^2 - (ND^2 + OD^2)}{2 OM} \\
 &= \frac{OM^2 - (OD \cdot MD + OD^2)}{2 OM} \\
 &= \frac{OM^2 - OD(MD + OD)}{2 OM} = \frac{OM^2 - OD \cdot OM}{2 OM} \\
 &= \frac{OM - OD}{2} = \frac{DM}{2} = DW
 \end{aligned}$$

Construction:

- 1) Draw diameter MC of O_1
- 2) Draw MN tangent to O_2 .
- 3) Draw radius ON to point of tangency.
- 4) Draw $ND \perp OM$.
- 5) Bisect DM at W.
- 6) DW is radius of required circle (proved in analysis)
- 7) Select any point I on O_2 and draw O_3 , with $DW = r$.
- 8) Select any point P on O_1 and draw tangents to O_3 cutting O_1 again at Q, S.
- 9) This construction fulfills requirement.

Discussion:

- 1) There can be an infinite number of points P selected to fill the necessary requirements, since an infinite number of triangles can be constructed with a given circumcircle and incircle (Alshiller-Court, Sec. 204).
- 2) An infinite number of circles O_3 may be constructed with center on O_2 and radius $= r$.

Also solved by J. Rosenbaum and the proposer.

No. 85. Proposed by Norman Anning, University of Michigan.

Simplify the expression:

$$1 + 2 \cos 2A + 2 \cos 4A + \dots + 2 \cos 2nA$$

n being a positive integer.

A note by the proposer:

The given expression has as its simpler form

$$\frac{\sin(2n+1)A}{\sin A}$$

When this is equated to zero we have a cyclotomic equation for dividing a given circumference into $(2n+1)$ equal parts. This equation is algebraic of degree n and has integral coefficients when $2 \cos 2A$ is chosen as the unknown.

Solution by R. A. Miller, University of Mississippi.

$$\begin{aligned} & 1 + 2 \cos 2A + 2 \cos 4A + 2 \cos 6A + \dots + 2 \cos 2nA \\ &= \frac{\sin A}{\sin A} = \frac{\sin 3A - \sin A}{\sin A} = \frac{\sin 5A - \sin 3A}{\sin A} + \dots \\ & \quad + \frac{\sin(2n+1)A - \sin(2n-1)A}{\sin A} \\ &= \frac{\sin(2n+1)A}{\sin A} \end{aligned}$$

Also solved by A. C. Briggs, Wilmington, Ohio; Paul K. Reese, Texas Tech; J. Rosenbaum, Hartford Federal College, Dewey C. Duncan, University of California; J. W. Faust, Central State Teachers College (Michigan).

No. 87. Proposed by F. A. Rickey, L. S. U.

A diameter AB of a certain circle is extended through A to C so that the length of AC is 3 inches. At B, a line segment BD is drawn perpendicular to AB. The length of BD is 9 inches. Determine the length of AB so that DC shall be tangent to the circle. Can this problem be solved without the use of algebraic equations of degree higher than the second?

Solutions to this problem were received from the following but none solved it without cubic equation:

W. B. Clarke, A. C. Briggs, E. N. Yeager, Toledo Teachers College, Gordon Walker, L. S. U.; J. W. Foust, and R. A. Miller.

No. 88. Proposed by Henry Schroeder, Louisiana Polytechnic Institute.

Is it possible with ruler and compasses to divide a triangle by a line from a vertex to the opposite side into two triangles whose incircles will be equal?

Solved by Norman Anning, University of Michigan.

Let AD divide triangle ABC in the required manner and use letters x, y, z to denote BD, DC, AD. It will be shown that x is a root of a quadratic equation whose coefficients are rational functions of a, b, c , the sides of the given triangle. As a consequence the point D is ruler-and-compasses constructible.

Because D is on BC,

$$x + y = a. \quad (1)$$

From the fact that the inradii of triangles ABD and ADC are equal,

$$\begin{aligned} \frac{2(\triangle ABD)}{c + x + z} &= \frac{2(\triangle ADC)}{b + y + z}, \\ \frac{c + x + z}{b + y + z} &= \frac{\triangle ABD}{\triangle ADC} = \frac{x}{y}, \\ cy - bx &= (x - y)z. \end{aligned} \quad (2)$$

Another equation involving z comes from equating the values of $\cos B$ as obtained from triangles ABD and ABC:

$$\frac{c^2 + x^2 - z^2}{2cx} = \frac{c^2 + a^2 - b^2}{2ca}.$$

This, by use of (1), may be written

$$yc^2 + xb^2 = a(xy + z^2). \quad (3)$$

Elimination of y and z from (1), (2), and (3) and dropping of two trivial solutions, $x=0$ and $x=a$, leads to

$$4ax^2 - 4x(a^2 + c^2 - b^2) + (a^3 + 3ac^2 - ab^2 - 2abc) = 0. \quad (4)$$

$$2ax = a^2 + c^2 - b^2 \pm (c - b)\sqrt{(c + b + a)(c + b - a)}.$$

Q. E. D.

PROBLEMS FOR SOLUTION

No. 93. Proposed by Dewey C. Duncan, University of California.

All equilateral hyperbolas which pass through 3 fixed points of the plane intersect in a definite fourth point. What geometrical relationship exists among these four points?

No. 94. Proposed by Walter B. Clarke, San Jose, Cal.

OA and OB are two radii of a circle and not in a straight line. On OA take OC, with C inside circle and not at midpoint of OA. On OB take BD=OC. With centers C and D and radii CA and BD, describe circles intersecting at E and F, taking E as the intersection that lies within the angle AOB. Show that angle EOF is a right angle.

No. 95. Proposed by Edmund N. Yeager, Toledo (Ohio) Teachers College:

A man enters a bank and has a check cashed. The teller mistakes the figures and pays cents for dollars and dollars for cents. The man then pays a bill for \$24.11 after which he finds he has twice as much money as the face of the original check. What was the face of the check? Show that the solution is unique.

No. 96. Proposed by Walter B. Clarke:

In a circle with center O, inscribe triangle ABC

Drop a perpendicular from A to D on BC

Drop a perpendicular from B to E on OA

Drop a perpendicular from C to E' on OA

Show that triangles ABC and DEE' are equiangular and that DE and DE' are both perpendicular to sides meeting at A.

No. 97. Proposed by J. Rosenbaum, Hartford Federal College.

Prove that if ABC is a triangle of maximum area inscribed in an ellipse (See problem No. 89), and if P is any point on the ellipse, then the sum of the squares of the areas of the triangles PAB, PBC, and PCA is equal to the square of the area of triangle ABC.

No. 98. Proposed by J. Rosenbaum:

Given a fixed triangle ABC, find the locus of a point P in space such that $(PAB)^2 + (PBC)^2 + (PCA)^2 = (ABC)^2$.



Book Reviews

Edited by
P. K. SMITH



Integrated Mathematics with Special Application to Analysis. By John A. Swenson. Edwards Brothers, Ann Arbor, Michigan. 1935—473 pages.

This text was written by a man teaching in both high school and in college. Mr. Swenson is head of the department of mathematics in the Wadleigh High School, New York City, and an Associate in Mathematics in Teachers College, Columbia University. One gathers immediately that the author is qualified to write a text from the viewpoint of both the high school and college. The above is one of a series of texts called "Integrated Mathematics." It is written for use in the twelfth grade.

An interesting feature of the text is its inclusion of analytic geometry and calculus in high school mathematics. Calculus is not taught as a separate topic but is weaved in "as the mortar or unifying element which is capable of binding together the various parts of algebra, geometry, and trigonometry," according to the author.

Chapter I covers briefly the function concept. Chapter II is devoted to the linear function. In this chapter the student is introduced to Cartesian geometry. A complete treatment, as is found in the average analytic geometry text, is given on the straight line. In this chapter the delta notation is introduced. The chapter concludes with a large list of rate problems which involve linear functions. The writer feels that only with a very selected group could rate problems be grasped at so early a stage.

Chapter III is devoted to "The number system of algebra." Real numbers are treated in an elementary manner. The author gives a lucid treatment of complex numbers by use of vectors. The proof of the addition formulas of trigonometry is given employing the complex number. The chapter ends with the fundamental formulas of trigonometry and De Moire's theorem.

In Chapter IV "The Polynomial Function" is treated. This chapter includes the material usually found in a college algebra on the theory of equations. This includes a short discussion on constructions with the ruler and compasses.

"The Conic Sections" is the subject of Chapter V. The usual material found in most analytical geometry texts is covered.

In Chapters VI, VII and VIII differentiation, integration and differentials are treated. In the first part of Chapter VI the theorems on limits are developed. Maxima and minima and rate problems are taken up in Chapter VI. The work in these chapters is handled from a critical approach.

Chapters IX, X, XI, XII, and XIII cover respectively "Exponential and Logarithmic Functions," "Permutations and Combinations," "Probability and Statistics," "Life Insurance," and "Determinants." The chapter on Probability and Statistics contains sixty-three pages.

This text is very interesting. It is written for the student who likes his mathematics; for one who has the will to dig down. It was surely written with a selected few in mind. With a good teacher and the right students the text could be used. The book is certainly well done and deserves close examination by teachers of twelfth grade mathematics. The style and the mechanical make-up are pleasing.

P. K. SMITH

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